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RELATIVELY CONVEX SUBGROUPS OF ORDERED GROUPS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled RELATIVELY CONVEX SUBGROUPS OF ORDERED GROUPS submitted by DONALD H. TOSH in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

In Chapter I, a brief background of ordered groups is given, including definitions and results that will be used in the following chapters. The second chapter deals with properties of ordered groups. The chain of convex subgroups of an ordered group is discussed at length, and Theorem 4, which is due to Rieger and Rodderyugin, has important applications to relatively convex subgroups. The concluding chapter deals with the questions of deciding when a subgroup is relatively convex. The question, "Is every strongly isolated infrainvariant subgroup of an ordered group relatively convex in that group?" was put forth by Kokorin, and although this question remains open, partial answers are given.

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CHAPTER I

In this chapter I present some basic definitions, terms, and results that will be used in this paper. This is done to avoid any confusion that may arise in the terminology.

Definition: The relation $\leq$ is a partial order on a set X if $\leq$ satisfies:

(1) $x \leq x$ (reflexive)

(2) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetric)

(3) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitive)

for all $x,y,z \in X$.

Definition: A group G is a partially ordered group if there is a relation $\leq$ on the set of elements of G satisfying:

(1) $\langle G,\leq\rangle$ is a partially ordered set

(2) if $a,b \in G$ and $a \leq b$ then for all $x,y \in G$ we have $xay \leq xby$, (that is, right and left multiplication preserve order).

There is another way of defining a partially ordered group. If G is a partially ordered group, then we can let P be the set defined as:

$$P = \{x \mid x \in G , e \leq x\}$$

where e is the identity element.

The set P is seen to have the following properties:

(1) $e \in P$

(2) if $x \in P$, $x \neq e$, then $x^{-1} \notin P$

(3) $PP \subseteq P$

(4) $x^{-1}Px \subseteq P$ for all $x \in G$.

These properties follow immediately from the definition of P . P is called the positive cone of G , and it defines the partial order on G by the identity:

$$a \leq b \quad \text{if and only if} \quad a^{-1}b \in P .$$

Proposition: A subset P is the positive cone of some partial order of G if and only if P satisfies:

(1) $P \cap P^{-1} = \{e\}$

(2) $PP \subseteq P$

(3) $x^{-1}Px \subseteq P$ for all $x \in G$.

In other words, any invariant subsemigroup P of G satisfying $P \cap P^{-1} = \{e\}$ defines a partial order on G .

Definition: A group G is ordered (linearly ordered, fully ordered) if the relation $\leq$ defines a full order on the set G . (That is, any two elements of G are comparable.) In terms of the positive cone P ; G is ordered if $P \cup P^{-1} = G$.

The partial order P on a group G induces a partial order on a subgroup H of G satisfying:

$$P(H) = H \cap P(G) \quad \text{where} \quad P(H)\ ,\ P(G)$$

are the positive cones of H , G respectively.

Definition: A subset X of a partially ordered set G is convex in G if X satisfies:

Given $a,b \in X$ and $c \in G$ such that $a \leq c \leq b$, then $c \in X$.

Definition: A convex subgroup of a partially ordered group G is a subgroup which is a convex subset of G .

Some elementary properties are:

(1) C is a convex subgroup of G if and only if $P(C)$ is a convex subset of $P(G)$.

(2) A convex subgroup of a convex subgroup is again convex in the whole group.

(3) The intersection of convex subgroups is again a convex subgroup.

A partially ordered group containing no proper normal convex subgroups is said to be o-simple (order-simple).

The factor group G/N of a partially ordered group G with respect to a normal convex subgroup N can be made into a partially ordered group by defining the order relation between the cosets by the rule:

$$aN \leq bN \quad \text{if and only if} \quad a' \leq b' \quad \text{for some} \quad a' \in aN ,\ b' \in bN .$$

An equivalent definition is $P(G/N)$ is the image of $P(G)$ under the natural homomorphism of G onto G/N. This partial order of G/N is called induced. Notice that the induced partial order on G/N can only be defined if N is convex.

Let $S(a_1,\cdots,a_n)$ denote the normal subsemigroup of a group G that is generated by the elements $a_1,\cdots,a_n$; and define $S'(a_1,\cdots,a_n)$ as $S(a_1,\cdots,a_n)$ with e adjoined. These normal subsemigroups play an important role in ordered groups. This is due to their following properties:

(1) if $a \in P$ then $S'(a) \subseteq P$

(2) if $a \in P$ and $a \neq e$ then $P \cap S(a^{-1}) = \phi$

(3) $S'(a_1,\cdots,a_n) = S'(a_1)\cdots S'(a_n)$

(4) $(S(a_1,\cdots,a_n))^{-1} = S(a_1^{-1},\cdots,a_n^{-1})$

where P is a partial order on the group.

The next result has numerous consequences:

Theorem 1 (Fuchs, [2], p. 34): A partial order P of a group G can be extended to a full order of G if and only if it has the property: For every finite set of elements $a_1,\cdots,a_n \in G$ $(a_i \neq e)$ the signs

$\epsilon_1, \cdots, \epsilon_n$ $(\epsilon_i = \pm 1)$ can be chosen such that $P \cap S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) = \phi$.

Corollary 1.1: A group G is orderable if and only if for all $a_1, \cdots, a_n \in G$ $(a_i \neq e)$ the signs $\epsilon_1, \cdots, \epsilon_n$ $(= \pm 1)$ can be chosen such that $e \notin S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n})$.

In abelian groups this simplifies to the following: An abelian group is orderable if and only if it is torsion free.

Definition: A partial order on G is isolated if for any $a \in G$, $a^n \geq e$ implies $a \geq e$ (equivalently, $a^n \in P$ implies $a \in P$).

A subgroup H of G is isolated if for any $a \in G$, $a^n \in H$ implies $a \in H$.

Definition: A partial order on G is strongly isolated if given any $a, x_1, \cdots, x_n \in G$, $(x^{-1}\ a\ x_1) \cdots (x_n^{-1}\ a\ x_n) \geq e$ implies $a \geq e$ (equivalently, $(x_1^{-1}\ a\ x_1) \cdots (x_n^{-1}\ a\ x_n) \in P$ implies $a \in P$).

A subgroup H of G is strongly isolated if given any $a, x_1, \cdots, x_n \in G$, $a(x_1^{-1}\ a\ x_1) \cdots (x_n^{-1}\ a\ x_n) \in H$ implies $a, (x_1^{-1}\ a\ x_1), \cdots, (x_n^{-1}\ a\ x_n) \in H$.

A group G is an R^*-group if the identity subgroup is strongly isolated.

Note: There is a difference between "strong isolation" and "strict isolation". For clarification, see Appendix.

Definition: A partial order Q is an extension of a partial order P if $P \subseteq Q$.

Definition: A group G is an 0^*-group (fully orderable) if every partial order on G can be extended to a full order.

Theorem 2 (Ohnishi, [2], p. 39): A group G is an 0^*-group if and only if G satisfies:

(1) if $b,c \in S(a)$, then $S(b) \cap S(c) \neq \phi$

(2) if $a \neq e$, then $e \notin S(a)$

for all $a,b,c \in G$.

Corollary 2.1: An abelian group is 0^* if and only if it is torsion-free.

Corollary 2.2: A factor group G/H of an 0^*-group is again 0^* if and only if it satisfies condition (2) of Theorem 2.

Three terms that will be used frequently in this paper are "infrainvariant", "jump", and of course "relatively convex". Infrainvariance is a generalization of normality.

Definition: A subgroup H of a group G is infrainvariant if for any $g \in G$ we have either $g^{-1}Hg \subseteq H$ or $H \subseteq g^{-1}Hg$.

Jumps are used when dealing with a chain of subgroups.

Definition: If $\sum$ is a chain of subgroups of a group G and $C,D \in \sum$ such that $C \subsetneq D$, then $C \subseteq D$ is a jump (denoted $C \prec D$) in $\sum$ if there are no other subgroups $H \in \sum$ satisfying $C \subsetneq H \subsetneq D$.

Definition: A subgroup H of an ordered group G is relatively convex in G if there is an ordering on G which makes H convex.

Finally, I would like to state a well known theorem. The object of this paper is to generalize this result.

Theorem 3: A normal subgroup N of an ordered group G is relatively convex in G if and only if G/N is orderable.

CHAPTER II

I would now like to discuss some properties of ordered groups (groups admitting a full order). I have already covered two methods of determining whether a group is orderable. These are:

(1) A group G is orderable if and only if for all $a_1, \cdots, a_n \in G$ $(a_i \neq e)$ the signs $\epsilon_1, \cdots, \epsilon_n$ $(\epsilon_i = \pm 1)$ can be chosen such that

$$e \notin S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) .$$

(2) A group G is orderable if and only if there is a normal subsemigroup P of G with the properties $P \cap P^{-1} = \{e\}$, $P \cup P^{-1} = G$.

There is a third important method of deciding whether a group is orderable. This involves the chain of convex subgroups. If G is an ordered group with order P, let $\sum$ be the set of all convex subgroups of G (convex with respect to P). $\sum$ can be seen to have the following properties ([2], p. 50):

(1) $\sum$ is a chain containing $\{e\}$ and G. If $\{C_\lambda\}$ $(\lambda \in \Lambda)$ is a subset of $\sum$, then $\bigcap_{\lambda\in\Lambda} C_\lambda$ and $\bigcup_{\lambda\in\Lambda} C_\lambda$ are in $\sum$.

(2) If $C \in \sum$ and $g \in G$, then $g^{-1}Cg \in \sum$.

(3) If $C \prec D$ is a jump in Σ, then C is normal in D and D/C is isomorphic to a subgroup of the real numbers.

(4) If $C \prec D$ is a jump, then $[N(C),N(C),D] \subseteq C$.

(5) If $C \in \Sigma$ and $S(a) \cap C \neq \phi$, then some conjugate of a lies in C.

One thing that should be made clear is some further properties which follow from those given. In particular, in the system Σ, if $C \prec D$ is a jump, then $C^g \prec D^g$ is also a jump for all $g \in G$. For if not, then there exists a $K \in \Sigma$ and a $g \in G$ such that $C^g \subsetneq K \subsetneq D^g$. But this implies that $(C^g)^{g^{-1}} \subsetneq K^{g^{-1}} \subsetneq (D^g)^{g^{-1}}$, and hence $C \subsetneq K^{g^{-1}} \subsetneq D$. Since $K \in \Sigma$, then $K^{g^{-1}} \in \Sigma$ and $C \subseteq D$ is therefore, by definition, not a jump. This contradicts our assumption that $C \prec D$ was a jump. Therefore, $C \prec D$ is a jump if and only if $C^g \prec D^g$ is a jump for all $g \in G$.

Now, if we assume that $g \in D$, then $D^g = D$ for all $g \in D$. Since $C^g \prec D^g$ must be a jump, then C^g must be equal to C for all $g \in D$. Therefore, C is normal in D.

Further, if $g \in N(D)$, then $D^g = D$ and hence $C^g = C$ for all $g \in N(D)$. Since $C^g = C$ we have $g \in N(C)$ and therefore $N(D) \subseteq N(C)$. Similarly we can show that $N(C) \subseteq N(D)$.

A system Σ of subgroups is called infrainvariant if for all $H \in \Sigma$ we have $H^g \in \Sigma$ for all $g \in G$; it is called complete if it is closed with respect to arbitrary unions and intersections.

Hence, if $\sum$ is a complete infrainvariant chain of subgroups, and $C \prec D$ is a jump in $\sum$, from above it follows that C is normal in D and $N(C) = N(D)$.

Theorem 4 (Rieger, Podderyugin, [2], p. 51): A group G is orderable if and only if it contains a system $\sum$ of subgroups satisfying conditions (1) - (5) above. Also, there is an order on G making all the subgroups in $\sum$ convex.

However, Fuchs proves this result with the intention of building an order on G which makes $\sum$ the set of all convex subgroups. These properties do not assure this, and as a result a stronger condition is introduced to replace conditions (4) - (5), and hence yielding Fuchs' desired result.

On the other hand, I wish to reduce the theorem to the case requiring the least number of conditions. But first I need the idea of Ω-orderability.

Definition: A group admitting a linear order that is invariant in the domain of operators Ω is said to be Ω-orderable. In particular, a subgroup H of a group G is G-orderable if it is Ω-orderable where Ω consists of all inner automorphisms of G. That is, H is G orderable if there is an order P on H such that $g^{-1}Pg \subseteq P$ for all $g \in G$.

Theorem 5 (Kokorin, [4]): A group G is orderable if and only if G contains a system $\sum$ of subgroups with the following properties:

(1) $\sum$ is a chain containing $\{e\}$ and G. If $\{C_\lambda\}$ $(\lambda\in\Lambda)$ is a subset of $\sum$, then $\bigcup_{\lambda\in\Lambda} C_\lambda$ and $\bigcap_{\lambda\in\Lambda} C_\lambda$ are in $\sum$.

(2) $g^{-1}Hg \in \sum$ for all $H \in \sum$, $g \in G$.

(3) If $C \prec D$ is a jump from $\sum$, then D/C is $N(C)/C$-orderable.

Also, there is an order P on G such that the elements of $\sum$ are convex in P.

Proof: Since the system of convex subgroups of G satisfy conditions (1) - (3) above, we need only show sufficiency.

Let $\sum$ be given satisfying the conditions of the theorem.

We can divide the jumps in $\sum$ into disjoint conjugacy classes by the rule:

$C \prec D$ is conjugate to $C' \prec D'$ if and only if there is a $g \in G$ such that $g^{-1}Cg = C'$, $g^{-1}Dg = D'$.

Let $H \prec H_{\alpha+1}$ be a jump in $\sum$. We can define an ordering P_α on $H_{\alpha+1}/H_\alpha$ such that P_α is invariant under all inner automorphisms of $N(H_\alpha)/H_\alpha$ (by hypothesis). This ordering can be imposed on all the jumps in the conjugacy class containing $H_\alpha \prec H_{\alpha+1}$ in the following way:

If $C = H_\alpha^g$, $D = H_{\alpha+1}^g$ then the ordering on D/C is defined to be P_α^g.

We must show that if there is $g,h \in G$ such that

$$C = H_\alpha^g = H_\alpha^h \ , \quad D = H_{\alpha+1}^g = H_{\alpha+1}^h \ , \quad \text{then} \quad P_\alpha^g = P_\alpha^h \ .$$

Assume not, then there is $\bar{p} \in P_\alpha$ such that $(\bar{p})^g \in (P_\alpha^h)^{-1}$. But this would mean that $(\bar{p})^{gh^{-1}} \in P_\alpha^{-1}$. Since $(H_\alpha)^{gh^{-1}} = H_\alpha$, $gh^{-1} \in N(H_\alpha)$ and hence conjugation by gh^{-1} is an inner automorphism of $N(H_\alpha)/H_\alpha$ and must fix P_α. This gives the necessary contradiction. Therefore, the imposition of P_α on all the jumps in its conjugacy class is well defined and does not depend on the element of conjugation.

Now, from each class of jumps we choose a representative whose order will determine the order on all the rest of the jumps in its class.

We can now define an order P on G. For each $g \in G$, let $C_g \prec D_g$ be the jump in $\sum$ determined by g.

$$(C_g = \cup\{C \mid g \notin C \ , \ C \in \textstyle\sum\} \ , \quad D_g = \cap\{D \mid g \in D \ , \ D \in \textstyle\sum\})$$

Put $g \in P$ if and only if $\bar{g} \in P_g$, where P_g is the order on D_g/C_g. We must show that P is actually an order on G.

(1) Obviously $e \in P$, and if $g \in P$ and $g \neq e$, then $g^{-1} \notin P$. Therefore $P \cap P^{-1} = \{e\}$.

(2) Let $g,h \in P$ where g,h determine $C_g \prec D_g$, $C_h \prec D_h$ respectively.

If $D_g = D_h$, then $\bar{g},\bar{h} \in P_g = P_h$ and hence $\bar{g}\bar{h} = \overline{gh} \in P_g$, which implies $gh \in P$.

If $D_g \neq D_h$ assume without loss of generality $D_g \subseteq C_h$ (this or $D_h \subseteq C_g$ must hold).

Since $h \in P$ we have $\overline{h} \in P_h$, where $\overline{h} = hC_h$. Since $g \in D_g \subseteq C_h$, $\overline{gh} = \overline{h} \in P_h$ and hence $gh \in P$. Therefore, $PP \subseteq P$.

(3) Let $g \in P$ where g determines $C_g \prec D_g$. Then $\overline{g} \in P_g$ and by construction $(\overline{g})^x \in P_g^x$ for all $x \in G$. Therefore $P^x \subseteq P$ for all $x \in G$.

(4) Finally, any $g \in G$ will determine some jump $C_g \prec D_g$ in $\sum$. Since the order P_g on D_g/C_g is a full order, either $\overline{g} \in P_g$ or $\overline{g} \in P_g^{-1}$. Therefore, $P \cup P^{-1} = G$.

Hence, P is a full order on G . The fact that H is convex under P for all $H \in \sum$ follows easily from the construction.

Corollary 5.1: Theorem 5 still holds if the following conditions are interchanged:

(3) If $C \prec D$ is a jump in $\sum$, then D/C is $N(C)/C$ - orderable.

(3') For all $C \in \sum$, $N(C)/C$ is orderable.

Proof: If C is a convex subgroup, then C is normal and convex in $N(C)$, and hence $N(C)/C$ is orderable by Theorem 3. Conversely, if $C \prec D$ is a jump in $\sum$, then $D/C \subseteq N(C)/C$ and hence it is $N(C)/C$ orderable.

CHAPTER III

I would now like to consider the question: Given an ordered group G and a subgroup H of G, what properties must H have to ensure that H is relatively convex in G ?

In the case that we are told that H is relatively convex, we can see that it has the following properties:

(1) H is infrainvariant.

This follows from the fact that the set $\sum$ of all convex subgroups forms a chain.

(2) H is strongly isolated.

To prove this, assume that $g(x_1^{-1}gx_1)(x_2^{-1}gx_2)\cdots(x_n^{-1}gx_n) \in H$ but $x_n^{-1}gx_n \notin H$ where $g, x_1, \cdots, x_n \in G$. Also assume without loss of generality that $g \geq e$ in the order making H convex. Since $x_n^{-1}gx_n \notin H$, then for all $h \in H$, $h < x_n^{-1}gx_n$. Also $g(x_1^{-1}gx_1)\cdots(x_{n-1}^{-1}gx_{n-1}) \geq e$ since $g \geq e$. Then $h < x_n^{-1}gx_n \leq e(x_n^{-1}gx_n) \leq g(x_1^{-1}gx_1)\cdots(x_{n-1}^{-1}gx_{n-1})(x_n^{-1}gx_n)$ from which it follows that $g(x_1^{-1}gx_1)\cdots(x_n^{-1}gx_n) \notin H$ which is a contradiction. Hence $(x_n^{-1}gx_n) \in H$ and thus $g(x_1^{-1}gx_1)\cdots(x_n^{-1}gx_n)(x_n^{-1}g^{-1}x_n) \in H$ and we repeat the process, obtaining the result by induction.

(3) $\overline{H} = \bigcup_{g \in G} H^g$ and $\underline{H} = \bigcap_{g \in G} H^g$ are normal, relatively convex subgroups of G.

(4) For all $a_1, \cdots, a_n \in G \sim H$ it is possible to find signs $\epsilon_1, \cdots, \epsilon_n$ ($\epsilon_i = 1$ or -1) such that $S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) \sim H$ is a semigroup.

The signs need only be chosen to make $a_i^{\epsilon_i} \geq e$ in the order making H convex.

Note: Condition (4) implies condition (2).

These conditions are all necessary but they are not all needed to prove sufficiency, as is shown in the following theorem:

Theorem 6 (Kokorin, [5]): A subgroup H of an ordered group G is relatively convex in G if and only if (1) H is infrainvariant and (4) for all $a_1, \cdots, a_n \in G \sim H$ there are signs $\epsilon_1, \cdots, \epsilon_n$ ($= \pm 1$) such that $S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) \sim H$ is semigroup.

Proof: Again we need only show sufficiency.

Let H satisfy the conditions of the theorem. We construct the system $\sum$ which consists of:

(a) $\{e\}$, G , and the set of all conjugates H^g of H .

(b) all possible unions and intersections of subgroups in (a).

$\sum$ is obviously a chain of subgroups and each subgroup in $\sum$ will have property (4) as stated in the theorem. This is because we need only choose the ϵ_i as we would for H . Moreover, if $C \in \sum$, then $N(C)/C$ is orderable. For since C satisfies condition (4) of the theorem, $N(C)/C$

satisfies the requirements of Corollary 1.1, making it orderable. Hence, by Corollary 5.1 all the subgroups in Σ are relatively convex. In particular, H is relatively convex.

Unfortunately, condition (4) is a very unwieldy tool to use in determining whether a subgroup is relatively convex. As a result, the following question has arisen: In an ordered group, if a subgroup is infrainvariant and strongly isolated, is it then relatively convex?

Kokorin, in [4], notes: It seems that infrainvariance and strong isolation are not enough for relative convexity of a subgroup in an orderable group, but we have no relevant example available.

Perhaps the reason for this is that this question is related directly to an older, more general, as yet unanswered question. It is known that every ordered group is an R^* - group. But this problem still remains: Is every R^* - group orderable? The relationship is embodied in the next result, a relationship Kokorin seems to have overlooked.

Theorem 7: Every R^*-group is orderable if and only if every strongly isolated infrainvariant subgroup of an ordered group is relatively convex in that group.

Proof:

Sufficiency: Assume every strongly isolated infrainvariant subgroup of an ordered group is relatively convex.

Let G be an R^*-group. G is then isomorphic to a factor group F/N of a free group F and a normal subgroup N of F . (Note: It is

generally known that every free group is orderable). N is infrainvariant, in fact normal, and is also strongly isolated since F/N is an R^*-group. Therefore, N is a normal relatively convex subgroup of F and hence F/N is orderable (Theorem 3).

Since F/N is isomorphic to G, we have G is orderable.

Necessity: Assume every R^*-group is orderable.

Let H be a strongly isolated infrainvariant subgroup of an ordered group G. We must show H is relatively convex.

To do this we again build the complete infrainvariant system $\sum$ in the following way:

$\sum$ consists of (1) $\{e\}$ and G

(2) all possible conjugates H^g of H

(3) all possible unions and intersections of the subgroups in (2).

Let C be an element of $\sum$. Since H is strongly isolated, C will also be strongly isolated. But if C is strongly isolated, then $N(C)/C$ will be an R^*-group, and hence by assumption, $N(C)/C$ is orderable (this also holds for $\{e\}$, since G is orderable).

Therefore, by Corollary 5.1, H is relatively convex. This completes the proof.

So, as it is, we have not asked a new question, we have only reworded an older one. One way of looking at it is this: the conditions imposed on an R^*-group as compared to an ordered group (Corollary 1.1) are very closely related to the condition of strong isolation as compared to condition (4) of Theorem 6. The problem seems to be one of extending the properties of the semigroup $S(a)$ to the semigroup $S(a_1, a_2, \cdots, a_n)$ generated by more than one element.

However, the above result does have other uses.

Corollary 7.1: Let $\mathcal{X}$ be a class of groups with the following properties:

(1) Every R^*-group G in $\mathcal{X}$ is orderable.

(2) If $G \in \mathcal{X}$, then every subgroup H of G is in $\mathcal{X}$.

(3) If $G \in \mathcal{X}$, and H is a normal subgroup of G, then $G/H \in \mathcal{X}$.

Then every infrainvariant strongly isolated subgroup of an ordered group in $\mathcal{X}$ is relatively convex.

Proof: This result follows directly from the necessity part of Theorem 7.

Example. Let $\mathcal{X}$ be the class of all hypercentral-by-metabelain groups. It is easy to see that this class satisfies conditions (1) and (2) of Corollary 7.1. Also, in [3], N. Gupta and A. Rhemtulla show that every hypercentral-by-metabelian R^*-group is orderable. Therefore, from Corollary 7.1 we have that every infrainvariant strongly isolated subgroup of an ordered hypercentral-by-metabelian group is relatively convex.

Another way of looking at the problem is this. It could be that in any ordered group, if an infrainvariant strongly isolated subgroup H is contained in a special subgroup of the group, then H is relatively convex.

In particular, a subgroup K of an ordered group G is said to be G-fully orderable if under any maximal partial order P on G, P restricted to K is linear on K (that is, $K \subseteq P \cup P^{-1}$). I would like to prove a result using this concept, but first I need:

Lemma 8: If P is a maximal partial order on an R^*-group G and if for any $b,c \in S(a)$ we have $S(b) \cap S(c)$ is not empty, then $a \in P \cup P^{-1}$.

Proof: Assume $a \notin P \cup P^{-1}$. If $S(a) \cap P^{-1} = \phi$, then a could be included in P. Since P is maximal, this cannot happen. Hence $S(a) \cap P^{-1} \neq \phi$. Similarly, $S(a) \cap P \neq \phi$. It follows then that there exist elements $b,c \in S(a)$ such that $b \in P$, $c \in P^{-1}$. But this implies $S(b) \subseteq P$ and $S(c) \subseteq P^{-1}$. Therefore $S(b) \cap S(c) \subseteq P \cap P^{-1} = \{e\}$. Since $b,c \neq e$, then $S(b) \cap S(c) = \phi$ since G is an R^*-group. This contradicts our hypothesis.

Lemma 9: H is a G-fully orderable subgroup of an R^*-group G if and only if for any $a \in H$ and $b,c \in S(a)$ we have $S(b) \cap S(c) \neq \phi$.

Proof:

Necessity: Assume there exist an $a \in H$ and $b,c \in S(a)$ such that $S(b) \cap S(c) = \phi$. Then $S'(b)S'(c^{-1})$ is a partial order on G. Extending

this to a maximal partial order P on G we find a cannot be included in P or P^{-1}, and hence H is not G-fully orderable.

Sufficiency: Assume that for any $a \in H$ we have $b,c \in S(a)$ implies $S(b) \cap S(c) \neq \phi$. By Lemma 8, $a \in P \cup P^{-1}$ for any maximal partial order P.

Theorem 10: Let G be an ordered group and let K be a normal, relatively convex, G-fully orderable subgroup of G. If H is an infrainvariant strongly isolated subgroup of G and $H \subseteq K$, then H is relatively convex in G.

Proof: Let G, K, and H be as in the theorem. By Theorem 6 it is sufficient to show that given any $a_1, \cdots, a_n \in G{\sim}H$ it is possible to find signs $\epsilon_1, \cdots, \epsilon_n$ $(= \pm 1)$ such that $S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}){\sim}H$ is a semigroup. Since H is strongly isolated this is obviously true for $n = 1$.

Assume that this property does not hold. That is, there are $a_1, \cdots, a_n \in G{\sim}H$ such that $S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}){\sim}H$ is not a semigroup for all choices of the ϵ_i.

I want to show that $S(a_1^{\epsilon_1}, \cdots, a_{n-1}^{\epsilon_{n-1}}){\sim}H$ is also never a semigroup.

We can see that at least one of the a_i is in K, for otherwise we can choose the ϵ_i such that $a_i^{\epsilon_i} \in P$ for all $1 \leq i \leq n$, where P is the order making K convex. Then $S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}) \cap K = \phi$ and hence $S(a_1^{\epsilon_1}, \cdots, a_n^{\epsilon_n}){\sim}H$ is a semigroup. Therefore, without loss of generality, assume $a_n \in K$.

Assume there are fixed signs $\epsilon_1,\cdots,\epsilon_{n-1}$ such that $S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}})\sim H$ is a semigroup.

Since $S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}},a_n)\sim H$ is not a semigroup, we can find elements $x_1,x_2 \in S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}},a_n)\sim H$ such that their product x_1x_2 is in H. Furthermore, x_1 and x_2 will have the form $x_1 = s_1b_1$, $x_2 = s_2b_2$ where $s_1,s_2 \in S'(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}})$ and $b_1,b_2 \in S'(a_n)$. Then

$$x_1x_2 = s_1b_1\ s_2b_2 = s_1s_2(b_1^{s_2}b_2) \in H .$$

If $s_1s_2 \in H$, it follows that $s_1,s_2 \in H$ since $S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}})\sim H$ is a semigroup. In such a case $b_1^{s_2}b_2 \in H$ and since H is strongly isolated we have $b_1^{s_2},b_2 \in H$, and hence $b_1,b_2 \in H$. This contradicts our assumption that $x_1,x_2 \notin H$. Thus $s_1s_2 \notin H$ and $b_1^{s_2}b_2 \notin H$. Rewriting s_1s_2 as s_1 and $b_1^{s_2}b_2$ as b_1 we have that $s_1b_1 \in H$ where $s_1 \in S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}})\sim H$ and $b_1 \in S(a_n)\sim H$.

Since $S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}},a_n^{-1})\sim H$ is also not a semigroup we could repeat the above procedure to obtain an x_2,b_2 with the properties:

$$s_2b_2 \in H \ ; \ s_2 \in S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}})\sim H \ ; \text{ and } \ b_2^{-1} \in S(a_n)\sim H .$$

Letting $s_1b_1 = h_1 \in H$ and $s_2b_2 = h_2 \in H$ we get that

$$s_1^{-1}h_1 \in S(a_n) \quad \text{and} \quad h_2^{-1}s_2 \in S(a_n) .$$

By Lemma 9, $S(s_1^{-1}h_1) \cap S(h_2^{-1}s_2) \neq \phi$ (since $a_n \in K$ and K is G-fully orderable). That is, for some $x_1,\cdots,x_m ; y_1,\cdots,y_r \in G$

$$(s_1^{-1}h_1)^{x_1}\cdots(s_1^{-1}h_1)^{x_m} = (h_2^{-1}s_2)^{y_1}\cdots(h_2^{-1}s_2)^{y_r} .$$

Let $t \in \{x_1,\cdots,x_m,y_1,\cdots,y_r\}$ such that $H^{x_i}, H^{y_j} \subseteq H^t$ for all $1 \leq i \leq m$, $1 \leq j \leq r$. Then

$$(s_1^{-1}h_1)^{x_1t^{-1}}\cdots(s_1^{-1}h_1)^{x_mt^{-1}} = (h_2^{-1}s_2)^{y_1t^{-1}}\cdots(h_2^{-1}s_2)^{y_rt^{-1}}$$

where $(h_1)^{x_it^{-1}}$, $(h_2^{-1})^{y_jt^{-1}} \in H$ for all i,j ; and at least one of the $(s_1^{-1})^{x_it^{-1}}$, $(s_2)^{y_jt^{-1}} \notin H$ (namely the x_i or y_j equal to t) .

Gathering terms we get

$$(s_1^{-1})^{w_1}\cdots(s_1^{-1})^{w_{m'}} h_1' = (s_2)^{z_1}\cdots(s_2)^{z_{r'}} h_2'$$

for some h_1' , $h_2' \in H$. (This is done by moving all terms in H to the right by conjugation)

$(s_1^{-1})^{w_1}\cdots(s_1^{-1})^{w_{m'}}$, $(s_2)^{z_1}\cdots(s_2)^{z_{r'}} \notin H$ (for if $s \notin H$ then $s^h \notin H$ for any $h \in H$) and h_1' , $h_2' \in H$.

Letting $(s_1')^{-1} = (s_1^{-1})^{w_1}\cdots(s_1^{-1})^{w_{m'}}$ and $s_2' = (s_2)^{z_1}\cdots(s_2)^{z_{r'}}$ we have that s_1' , $s_2' \in S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}})\sim H$ but $s_1'\ s_2' \in H$. This contradicts the assumption that $S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}})\sim H$ is a semigroup.

Therefore, if $S(a_1^{\epsilon_1},\cdots,a_n^{\epsilon_n})\sim H$ is never a semigroup, then $S(a_1^{\epsilon_1},\cdots,a_{n-1}^{\epsilon_{n-1}})\sim H$ is also never a semigroup, assuming that $a_n \in K\sim H$.

Again at least one of the $a_1,\cdots,a_{n-1}$ must be in K and without loss of generality we may assume it is a_{n-1} . We can therefore repeat the above procedure and reduce the case to $n = 1$, which provides the final contradiction. This proves the theorem.

Unfortunately this proof is long and tedious but I have not found a preferable one. Kokorin proves it for the more restricted case of 0^*-groups (using a different method, see [4] and [6]), and then states that the proof of this result is the same.

Corollary 10.1: An infrainvariant strongly isolated subgroup of an 0^*-group G is relatively convex.

Proof: G is a normal, relatively convex, G-fully orderable subgroup of itself.

Corollary 10.2: In an ordered group G , any infrainvariant strongly isolated subgroup that is contained in the hypercentre H of G is relatively convex.

Proof: In [1] Fuchs shows that the hypercentre is G-fully orderable and relatively convex. It is also, of course, normal.

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APPENDIX

As was noted earlier, there is a difference between the terms "strong isolation" and "strict isolation" and I do not wish the two terms to be confused.

Definition: A subgroup H of a group G is said to be strictly isolated if for all $g, x_1, \cdots, x_n \in G$ such that

$$g^{x_1} g^{x_2} \cdots g^{x_n} \in H$$

we have that some conjugate g^x of g is in H for some $x \in G$.

A subgroup H of G is strongly isolated if for all $g, x_1, \cdots, x_n \in G$ such that

$$g\, g^{x_1} \cdots g^{x_n} \in H$$

we have that

$$g, g^{x_1}, \cdots, g^{x_n} \in H .$$

Strong isolation is clearly the stronger of the two conditions. In the case that H is normal or relatively convex they turn out to be equivalent. In general, however, this does not happen.

Example: Let G be the group on two generators defined by

$$G = \langle a, b \; ; \; a^{-1} b a = b^2 \rangle$$

and define H to be the subgroup generated by b^4. G is orderable and in fact H is infrainvariant; however H is not strongly isolated since $(a^{-1}ba)(a^{-1}ba) = b^4 \in H$ but $a^{-1}ba = b^2 \notin H$. Furthermore, H is strictly isolated since a conjugate of b, namely $a^{-2}ba^2 = b^4$, lies in H.

Therefore, the two definitions are not equivalent.

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